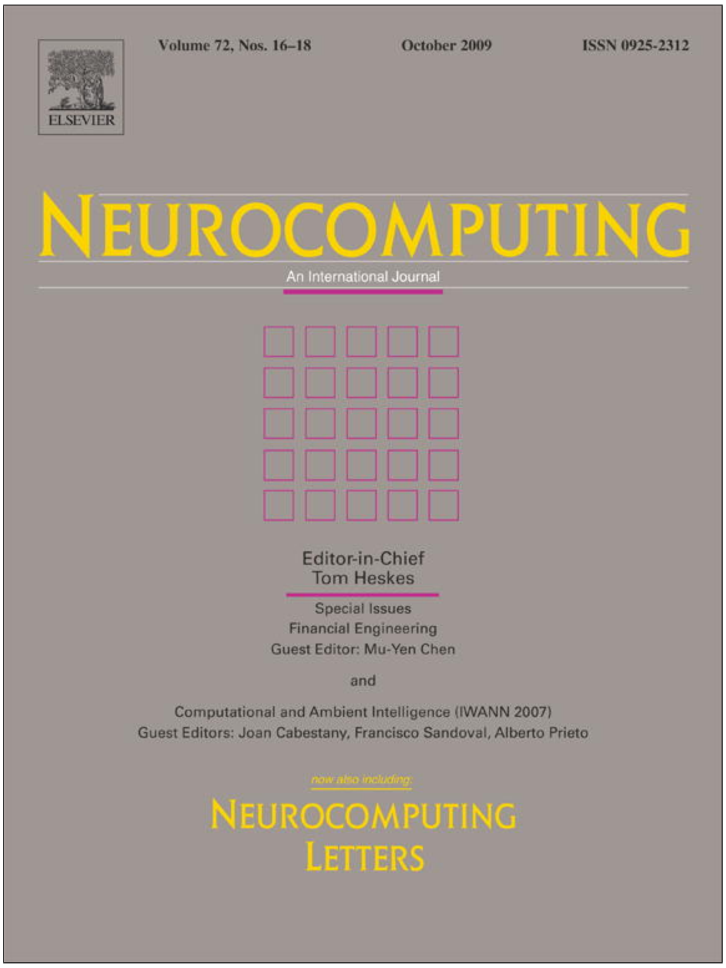
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A class of discrete time recurrent neural networks with multivalued neurons

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ABSTRACT

This paper discusses a class of discrete time recurrent neural networks with multivalued neurons (MVN), which have complex-valued weights and an activation function defined as a function of the argument of a weighted sum. Complementing state-of-the-art of such networks, our research focuses on the convergence analysis of the networks in synchronous update mode. Two related theorems are presented and simulation results are used to illustrate the theory.

1. Introduction

The multivalued neuron (MVN) was first introduced in [8] based on the principles of multivalued threshold logic over the field of complex numbers. Its theory was further extended in [11,6]. The main property of MVN neural networks is that the activation function maps complex-valued inputs into outputs lying on the unit circle. Therefore, MVN learning can be reduced to the movement along the unit circle. Another important property is that the functionality of an MVN is higher than the functionality of a sigmoidal neuron [7]. It has been shown that the multilayer neural network based on MVN (MLMVN) outperforms a classical multilayer feedforward network and several kernel-based networks in terms of learning speed and network complexity [4]. As a result, successful applications of the MVN have been developed in recent years [1,3,4].

For MVN neural networks, there exist two main approaches to analysis and utilization. The first one takes advantage of error–correction rule (or the complex least mean square algorithm [14]) that minimizes the cumulative cycle error between the real network output and the target output by adjusting network parameters in the hidden and output layers. In fact, these networks are more like universal approximators. Some applications include nonlinear filters [2,5], classifiers [1,4], and Mackey–Glass time series prediction [3]. Although most of MVN networks noted above are feedforward neural networks, such kind of error–correction approach is also applicable to other complex-valued recurrent neural networks [9,10].

The second approach is to treat complex-valued neural network as a recurrent dynamic system able to acquire some useful properties without supervised training. These approaches are often referred to as associative memory design [6,7,12,13]. While the stability analysis and qualitative analysis methods are popular and well developed in real-valued recurrent dynamic neural networks, research into complex-valued networks as dynamic system has not progressed equally fast. In [12], the stability of a Hopfield MVN-based neural network is discussed, which establishes that a complex-valued MVN network in asynchronous update mode will be convergent if its weight matrix is Hermitian with nonnegative diagonal entries. A related associative memory method is proposed by employing inequalities in the network design [13].

In this paper, we study a class of discrete-time recurrent neural networks with multivalued neurons in synchronous update mode. Based on the energy minimization performed by the proposed network, we first prove that if connection weight matrix is Hermitian, then each trajectory of the network will converge to an equilibrium or periodic points with period of 2. Furthermore, if there exists a diagonal positive definite matrix D such that DW is a symmetric positive definite matrix, then the network is completely convergent. Simulation results are used to illustrate the theory.

The rest of this paper is organized as follows: the architecture of the proposed multivalued recurrent neural networks is described in Section 2. In Section 3, a theoretical analysis of the network is given, which includes two theorems of the proposed network. Simulations and illustrative examples are presented in Section 4. Conclusions are given in Section 5.

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2. Multivalued recurrent neural networks

The MVN model is based on the complex-signum operation (see Fig. 1). For a specified number of values K, called the resolution factor, and an arbitrary complex number u, the complex-signum function is defined as follows:

$$CSIGN(u) \triangleq \begin{cases} z^0, 0 \le \arg(u) < \varphi_0 \\ z^1, \varphi_0 \le \arg(u) < 2\varphi_0 \\ \vdots \\ z^{K-1}, (K-1)\varphi_0 \le \arg(u) < K\varphi_0 \end{cases}, \quad (1)$$

where φ_0 is a phase quantum delimited by K: $\varphi_0 = 2\pi/K$, and z is the corresponding Kth root of one: $z = e^{i\varphi_0}$. Then, the output state of each neuron is represented by a complex number from the set $\{z^0, z^1, \ldots, z^{K-1}\}$. For simplicity, in this article, we use $\sigma(\cdot)$ instead of $CSIGN(\cdot)$.

In our model, every two neurons i,j are coupled via their synaptic connection w_{ij} represented by a complex number. And each input $I_i(k+1)$ of the ith neuron is dependent upon the network state through synaptic weight

$$I_i(k+1) = \sum_{j=1}^{N} w_{ij} s_j(k) + h_i.$$

And the output of the ith neuron is defined as

$$s_i(k+1) = CSIGN(I_i(k+1) \cdot z^{1/2}),$$

where $z^{1/2} = e^{i(\varphi_0/2)}$ (see Fig. 2).

The equivalent vector form of network is defined as

$$S(k+1) = \mathrm{CSIGN}(WQS(k) + QH) \quad (2)$$

for $k \ge 0$, where $S(k)$, H are $n \times 1$ vectors

$$S(k) = [s_1(k), \ldots, s_N(k)]^T,$$

$$H = [h_1, \ldots, h_N]^T,$$

and Q is a diagonal matrix

$$Q = diag(z^{1/2}, \ldots, z^{1/2}) \triangleq \begin{bmatrix} z^{1/2} & & & \\ & z^{1/2} & & \\ & & \ddots & \\ & & & z^{1/2} \end{bmatrix}.$$

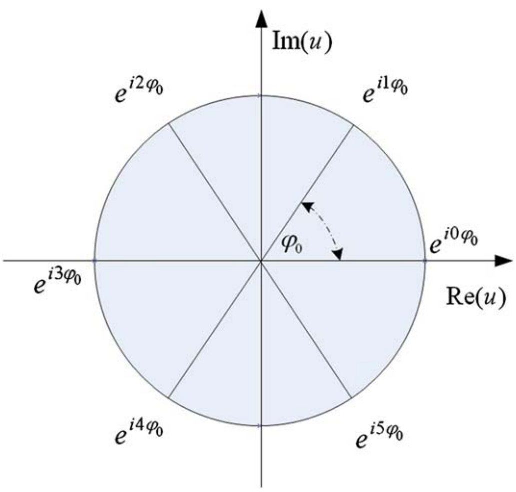

Fig. 1. Complex-signum function CSIGN (case shown for $K = 6$).

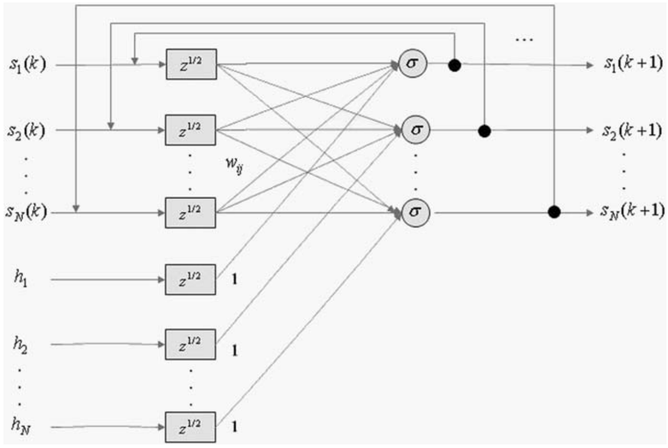

Fig. 2. Multivalued recurrent neural networks.

Note that for W being invertible, network (2) is dynamically equivalent to the network

$$x(k+1) = WQ\sigma(x(k)) + QH \quad (3)$$

by using

$$x_i(k+1) = I_i(k) \cdot z^{1/2}.$$

For convenience, in the following sections, we analyze the properties of network (3) directly.

3. Properties

Firstly, we provide preliminaries which will be used in the following to establish the theory.

For any $c \in \mathbb{C}$, we denote

$$c^* = (\overline{c})^T,$$

where $\overline{c}$ is the conjugate of c.

Definition 1. A vector x^* is called an equilibrium point (fixed point) of network (3), if it satisfies

$$x^* = WQ\sigma(x^*) + QH.$$

Denote by Ω to the set of equilibrium points of network (3).

Definition 2. An equilibrium point x^* is said to be stable (in the sense of Lyapunov) if the following statement is true: for every $\varepsilon > 0$, there exists $\delta > 0$ such that every solution $x(k)$ with $\|x(0) - x^*\| < \delta$ exists for all $k \ge 0$ and satisfies the inequality $\|x(k) - x^*\| < \varepsilon$ for $k \ge 0$. The norm $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^n$ or $\mathbb{C}^n$.

Definition 3. Network (3) is said to be completely convergent (completely stable), if each trajectory $x(k)$ satisfies

$$dist(x(k), \Omega) \triangleq \min_{x^* \in \Omega} \|x(k) - x^*\| \to 0$$

as $k \to +\infty$.

Lemma 1. *According to network* (3), *we define*

$$\Delta M_i(k+1) = z^{-1/2} \int_{x_i(k)}^{x_i(k+1)} [\sigma^*(s) - \sigma^*(x_i(k))]\, ds, \quad (4)$$

then it holds that

$$Re(\Delta M_i(k+1)) \geq 0$$

for all $i = 1, \ldots, N$ *and* $k \geq 0$.

Proof. Now, we suppose that there is a partition

$$\{x_i(k), \delta_1, \delta_2, \ldots, \delta_m, x_i(k+1)\}$$

in $[x_i(k), x_i(k+1)]$ according to their phase angles. Here, we define

$$\begin{cases} \sigma(\delta_j) = \delta_j \\ \delta_{j+1} = z\delta_j \end{cases}$$

for $j = 0, \ldots, m$, and $0 \leq m \leq K-1$. Especially, we define

$$\begin{cases} \sigma(x_i(k)) = \delta_0 \\ \sigma(x_i(k+1)) = \delta_m = z^m \delta_0 \end{cases}. \tag{5}$$

And $x_i(k+1)$ can also be defined as

$$x_i(k+1) = \Theta_i \delta_m e^{i\Delta\varphi} z^{1/2}, \tag{6}$$

where Θ_i represents the magnitude: $\Theta_i \geq 0$, and $\Delta\varphi$ represents the phase shift: $0 \leq \Delta\varphi < (\varphi_0/2)$.

Before calculating, we need to define the integral form

$$\int_{\delta_{j-1}}^{\delta_j} [\sigma^*(s) - \sigma^*(x_i(k))]\, ds \triangleq \lim_{\Delta\sigma \to (\delta_j)^-} \int_{\delta_{j-1}}^{\Delta\sigma} [\sigma^*(s) - \sigma^*(x_i(k))]\, ds$$

for $j = 1, \ldots, m$, where $\Delta\sigma \to (\delta_j)^-$ means that $\Delta\sigma$ approaches δ_j along the left-hand direction of δ_j. This definition is very similar to the definition of the left-hand limit in real analysis.

Then, by Eqs. (4)–(6), we have

$$\begin{aligned}
&\Delta M_i(k+1) \\
&= z^{-1/2} \int_{x_i(k)}^{x_i(k+1)} [\sigma^*(s) - \sigma^*(x_i(k))]\, ds \\
&= z^{-1/2} \left\{ \int_{x_i(k)}^{\delta_1} [\sigma^*(s) - \sigma^*(x_i(k))]\, ds + \int_{\delta_1}^{\delta_2} [\sigma^*(s) - \sigma^*(x_i(k))]\, ds \right. \\
&\quad \left. + \cdots + \int_{\delta_{m-1}}^{\delta_m} [\sigma^*(s) - \sigma^*(x_i(k))]\, ds + \int_{\delta_m}^{x_i(k+1)} [\sigma^*(s) - \sigma^*(x_i(k))]\, ds \right\} \\
&= z^{-1/2} [(\delta_1^* - \delta_0^*)(\delta_2 - \delta_1) + \cdots + (\delta_{m-1}^* - \delta_0^*)(\delta_m - \delta_{m-1}) \\
&\quad + (\delta_m^* - \delta_0^*)(x_i(k+1) - \delta_m)] \\
&= z^{-1/2} \{[\delta_1^*(\delta_2 - \delta_1) + \cdots \\
&\quad + \delta_{m-1}^*(\delta_m - \delta_{m-1})] - [\delta_0^*(\delta_2 - \delta_1) + \cdots + \delta_0^*(\delta_m - \delta_{m-1})] \\
&\quad - (\delta_m^* - \delta_0^*)\delta_m + (\delta_m^* - \delta_0^*) x_i(k+1)\} \\
&= z^{-1/2} \{[(z-1) + \cdots + (z-1)] - \delta_0^*(\delta_m - \delta_1) - (\delta_m^* - \delta_0^*)\delta_m \\
&\quad + (\delta_m^* - \delta_0^*) x_i(k+1)\} \\
&= z^{-1/2} \{[(z-1) + \cdots + (z-1)] + (z-1) + (\delta_m^* - \delta_0^*) x_i(k+1)\} \\
&= z^{-1/2} [m(z-1) + (\delta_m^* - \delta_0^*) \Theta_i \delta_m e^{i\Delta\varphi} z^{1/2}] \\
&= m(z^{1/2} - z^{-1/2}) + \Theta_i (1 - z^m) e^{i\Delta\varphi}.
\end{aligned}$$

Clearly, $Re(m(z^{1/2} - z^{-1/2})) = 0$. And we have

$$\begin{aligned}
Re(\Delta M_i(k+1)) &= Re(\Theta_i (1 - z^m) e^{i\Delta\varphi}) \\
&= \Theta_i (\cos(\Delta\varphi) - \cos(m\varphi_0 + \Delta\varphi)) \geq 0
\end{aligned}$$

for all $0 \leq m \leq K-1$.

This completes the proof. □

Theorem 1. *If* W *is a Hermitian matrix, then each trajectory of network* (3) *will converge to an equilibrium or periodic points with period of* 2.

Proof. Constructing the following energy function:

$$\begin{aligned}
E(k) &= -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \sigma(x_j(k)) \sigma^*(x_i(k+1)) \\
&\quad - \frac{1}{2} \left(\sum_{i=1}^N \sum_{j=1}^N w_{ij} \sigma(x_j(k)) \sigma^*(x_i(k+1)) \right)^* \\
&\quad - \frac{1}{2} \sum_{i=1}^N h_i (\sigma^*(x_i(k+1)) + \sigma^*(x_i(k))) \\
&\quad - \frac{1}{2} \left(\sum_{i=1}^N h_i (\sigma^*(x_i(k+1)) + \sigma^*(x_i(k))) \right)^* \\
&= -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \sigma(x_j(k)) \sigma^*(x_i(k+1)) \\
&\quad - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N w_{ij}^* \sigma^*(x_j(k)) \sigma(x_i(k+1)) \\
&\quad - \frac{1}{2} \sum_{i=1}^N h_i (\sigma^*(x_i(k+1)) + \sigma^*(x_i(k))) \\
&\quad - \frac{1}{2} \left(\sum_{i=1}^N h_i (\sigma^*(x_i(k+1)) + \sigma^*(x_i(k))) \right)^* \\
&= -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \sigma(x_j(k)) \sigma^*(x_i(k+1)) \\
&\quad - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \sigma^*(x_i(k)) \sigma(x_j(k+1)) \\
&\quad - \frac{1}{2} \sum_{i=1}^N h_i (\sigma^*(x_i(k+1)) + \sigma^*(x_i(k))) \\
&\quad - \frac{1}{2} \left(\sum_{i=1}^N h_i (\sigma^*(x_i(k+1)) + \sigma^*(x_i(k))) \right)^*.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\Delta E &= E(k+1) - E(k) \\
&= -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \sigma(x_j(k+1)) \sigma^*(x_i(k+2)) \\
&\quad - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \sigma^*(x_i(k+1)) \sigma(x_j(k+2)) \\
&\quad + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \sigma(x_j(k)) \sigma^*(x_i(k+1)) \\
&\quad + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \sigma^*(x_i(k)) \sigma(x_j(k+1)) \\
&\quad - \frac{1}{2} \sum_{i=1}^N h_i (\sigma^*(x_i(k+2)) - \sigma^*(x_i(k))) \\
&\quad - \frac{1}{2} \left(\sum_{i=1}^N h_i (\sigma^*(x_i(k+2)) - \sigma^*(x_i(k))) \right)^* \\
&= -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \sigma(x_j(k+1)) (\sigma^*(x_i(k+2)) - \sigma^*(x_i(k))) \\
&\quad - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \sigma(x_i^*(k+1)) (\sigma(x_j(k+2)) - \sigma(x_j(k)))
\end{aligned}$$

$$-\frac{1}{2}\sum_{i=1}^{N} h_i(\sigma^*(x_i(k+2)) - \sigma^*(x_i(k)))$$
$$-\frac{1}{2}\left(\sum_{i=1}^{N} h_i(\sigma^*(x_i(k+2)) - \sigma^*(x_i(k)))\right)^*$$
$$= -\frac{1}{2}\sum_{i=1}^{N} \sigma(x_i(k+2))(\sigma^*(x_i(k+2)) - \sigma^*(x_i(k)))z^{-1/2}$$
$$-\frac{1}{2}\left(\sum_{i=1}^{N} \sigma(x_i(k+2))(\sigma^*(x_i(k+2)) - \sigma^*(x_i(k)))z^{-1/2}\right)^*$$
$$= -\frac{1}{2}\sum_{i=1}^{N}(\Delta E)_i - \frac{1}{2}\sum_{i=1}^{N}((\Delta E)_i)^*$$
$$= -\sum_{i=1}^{N} Re((\Delta E)_i).$$

Here, we denote

$$(\Delta E)_i = \sigma(x_i(k+2))(\sigma^*(x_i(k+2)) - \sigma^*(x_i(k)))z^{-1/2}.$$

We suppose that

$$\sigma(x_i(k+2)) = \sigma(x_i(k))z^m,$$

where $0 \le m \le K-1$. Then, we have

$$Re(\Delta E)_i = Re((1-z^m)z^{-1/2}) = \cos(-\tfrac{1}{2}\varphi_0) - \cos(m\varphi_0 - \tfrac{1}{2}\varphi_0) \ge 0$$

for all $0 \le m \le K-1$.

Therefore,

$$\Delta E = -\sum_{i=1}^{N} Re((\Delta E)_i) \le 0,$$

which means that $E(k)$ is monotonously decreasing.

And, if $\Delta E = 0$, we have $m = 0$, which means

$$\sigma(x_i(k+2)) = \sigma(x_i(k))z^m = \sigma(x_i(k))$$

for all $i = 1, \ldots, N$. Therefore, if $\sigma(x_i(k+1)) = \sigma(x_i(k))$ for all $i = 1, \ldots, N$, clearly, each trajectory of the network converges to an equilibrium point, otherwise the trajectory converges to periodic points with periods of 2.

This completes the proof. □

Theorem 2. *If there exists a diagonal positive definite matrix D such that DW is a symmetric positive definite matrix, then network* (3) *is completely convergent.*

Proof. Suppose $D = diag(d_1, \ldots, d_N)$.

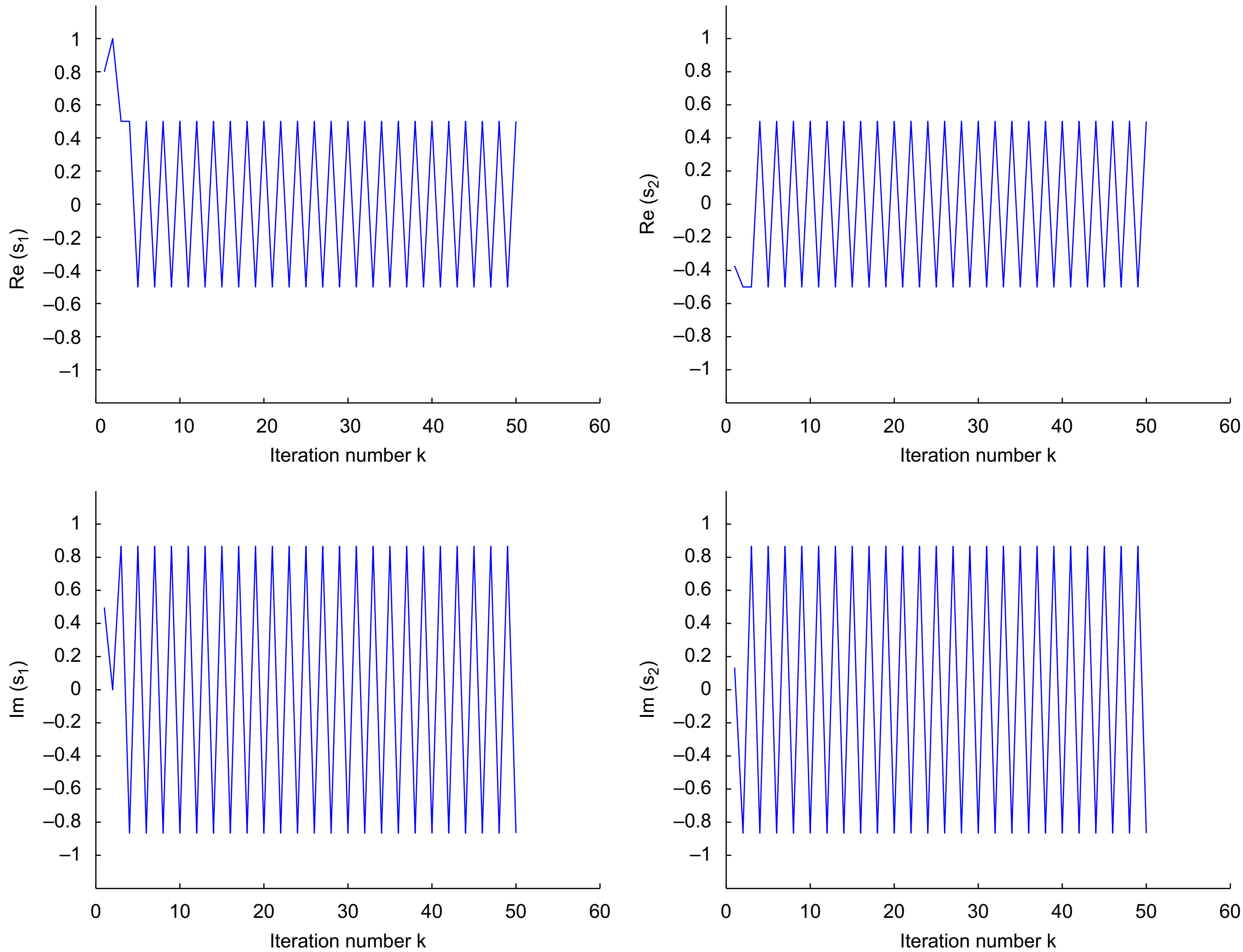

Fig. 3. Periodic trajectory of network (10) according to the activities of s_1, s_2.

Constructing the following energy function:

$$
\begin{aligned}
E(k) = &-\frac{1}{2}\sigma^*(x(k))DW\sigma(x(k)) \\
&-\frac{1}{2}\sigma^*(x(k))DH + \frac{1}{2}(Q\sigma(x(k)))^*Dx(k) \\
&-\frac{1}{2}z^{-1/2}\sum_{i=1}^{N} d_i \int_0^{x_i}(k)\sigma^*(s)\,ds \\
&+\left(-\frac{1}{2}\sigma^*(x(k))DH + \frac{1}{2}(Q\sigma(x(k)))^*Dx(k)\right)^* \\
&+\left(-\frac{1}{2}z^{-1/2}\sum_{i=1}^{N} d_i \int_0^{x_i}(k)\sigma^*(s)\,ds\right)^*.
\end{aligned}
$$

Clearly, the energy functional $E(k)$ is real-valued as long as the synaptic matrix W is a Hermitian matrix.

Then, we have

$$
\begin{aligned}
\Delta E &= E(k+1) - E(k) \\
&= -\frac{1}{2}(Q\sigma(x(k+1)) - Q\sigma(x(k)))^*DW(Q\sigma(x(k+1)) - Q\sigma(x(k))) \\
&\quad -\frac{1}{2}(Q\sigma(x(k+1)))^*DW(Q\sigma(x(k))) \\
&\quad -\frac{1}{2}(Q\sigma(x(k)))^*DW(Q\sigma(x(k+1))) \\
&\quad +\frac{1}{2}\sigma^*(x(k))DW\sigma(x(k)) + \frac{1}{2}\sigma^*(x(k))DW\sigma(x(k)) \\
&\quad -\frac{1}{2}(\sigma^*(x(k+1)) - \sigma^*(x(k)))DH + \frac{1}{2}(Q\sigma(x(k+1)))^*Dx(k+1) \\
&\quad -\frac{1}{2}(Q\sigma(x(k)))^*Dx(k) - \frac{1}{2}z^{-1/2}\sum_{i=1}^{N} d_i \int_{x_i(k)}^{x_i(k+1)} \sigma^*(s)\,ds \\
&\quad +\left(-\frac{1}{2}(\sigma^*(x(k+1)) - \sigma^*(x(k)))DH + \frac{1}{2}(Q\sigma(x(k+1)))^*Dx(k+1)\right)^* \\
&\quad +\left(-\frac{1}{2}(Q\sigma(x(k)))^*Dx(k) - \frac{1}{2}z^{-1/2}\sum_{i=1}^{N} d_i \int_{x_i(k)}^{x_i(k+1)} \sigma^*(s)\,ds\right)^* \\
&= -\frac{1}{2}Z^*(k+1)DWZ(k+1) + M(k+1) + M^*(k+1) \\
&= -\frac{1}{2}Z^*(k+1)DWZ(k+1) + 2Re(M(k+1)) \qquad (7)
\end{aligned}
$$

for all $k \geq 0$, and here we denote

$$Z(k+1) = Q\sigma(x(k+1)) - Q\sigma(x(k)),$$

and

$$
\begin{aligned}
M(k+1) &= -\frac{1}{2}(Q\sigma(x(k+1)))^*DW(Q\sigma(x(k))) + \frac{1}{2}\sigma^*(x(k))DW\sigma(x(k)) \\
&\quad -\frac{1}{2}(\sigma^*(x(k+1)) - \sigma^*(x(k)))DH + \frac{1}{2}(Q\sigma(x(k+1)))^*Dx(k+1) \\
&\quad -\frac{1}{2}(Q\sigma(x(k)))^*Dx(k) - \frac{1}{2}z^{-1/2}\sum_{i=1}^{N} d_i \int_{x_i(k)}^{x_i(k+1)} \sigma^*(s)\,ds
\end{aligned}
$$

for all $k \geq 0$.

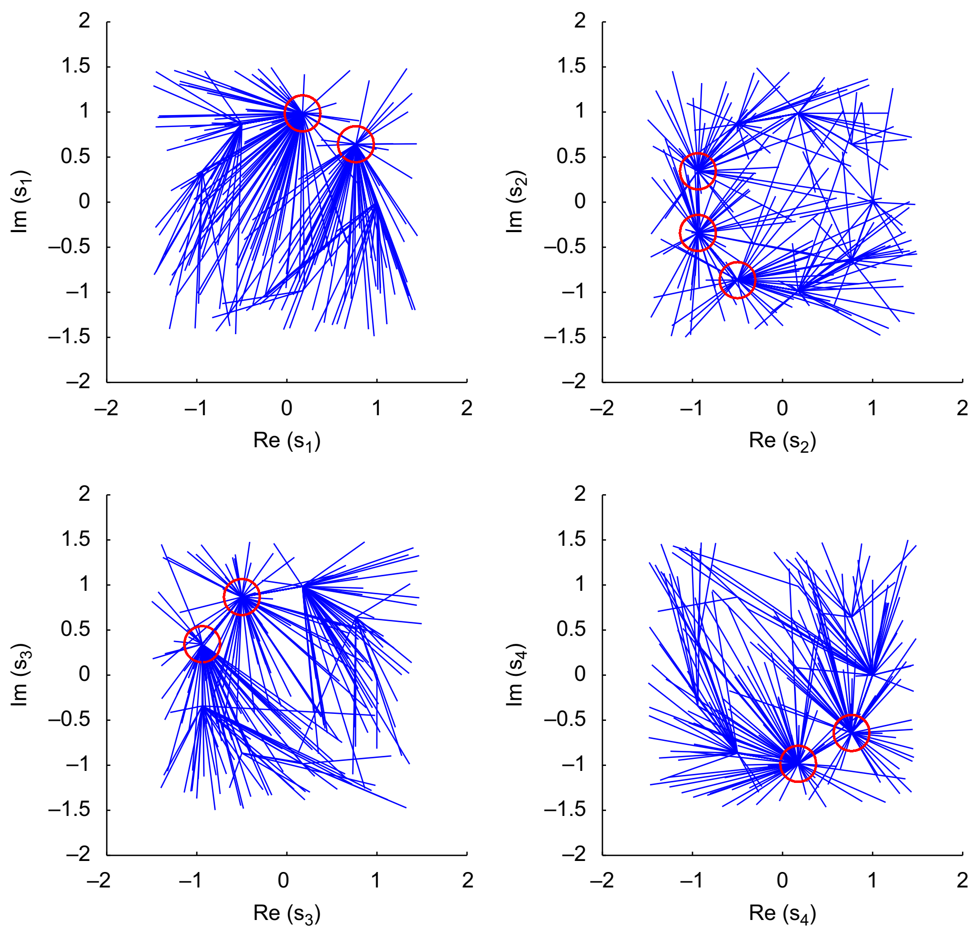

Fig. 4. Global attractivity of network (11), the attractive regions are marked with a circle according to the activities of s_1–s_4. There are nine states for each neuron ($K = 9$).

Since DW is a symmetric positive definite matrix,

$$-\tfrac{1}{2}Z^*(k+1)DWZ(k+1)\leq 0 \tag{8}$$

for all $k\geq 0$.

In order to make the change of energy function $\Delta E\leq 0$, the term $M(k+1)$ needs to be located in the left half-plane. And we have

$$\begin{aligned}M(k+1) &= -\frac{1}{2}(Q\sigma(x(k+1)))^*Dx(k+1)+\frac{1}{2}(Q\sigma(x(k)))^*Dx(k+1)\\ &\quad+\frac{1}{2}(Q\sigma(x(k+1)))^*Dx(k+1)-\frac{1}{2}(Q\sigma(x(k)))^*Dx(k)\\ &\quad-\frac{1}{2}z^{-1/2}\sum_{i=1}^{N}d_i\int_{x_i(k)}^{x_i(k+1)}\sigma^*(s)\,ds\\ &=\frac{1}{2}(Q\sigma(x(k)))^*Dx(k+1)-\frac{1}{2}(Q\sigma(x(k)))^*Dx(k)\\ &\quad-\frac{1}{2}z^{-1/2}\sum_{i=1}^{N}d_i\int_{x_i(k)}^{x_i(k+1)}\sigma^*(s)\,ds\\ &=-\frac{1}{2}z^{-1/2}\sum_{i=1}^{N}d_i\int_{x_i(k)}^{x_i(k+1)}[\sigma^*(s)-\sigma^*(x_i(k))]\,ds\\ &=-\frac{1}{2}\sum_{i=1}^{N}d_i\Delta M_i(k+1)\end{aligned}$$

for all $k\geq 0$.

By Lemma 1, we have

$$Re(M(k+1))=Re\left(-\frac{1}{2}\sum_{i=1}^{N}d_i\Delta M_i(k+1)\right)\leq 0 \tag{9}$$

for all $k\geq 0$.

From Eqs. (7)–(9), we have

$$\Delta E=-\tfrac{1}{2}Z^*(k+1)DWZ(k+1)+2\,Re(M(k+1))\leq 0.$$

Therefore, $E(k)$ is monotonously decreasing. Especially, if $\Delta E=0$, we have $m=0$, which means

$$\sigma(x_i(k+1))=z^m\delta_0=\sigma(x_i(k))$$

for all $i=1,\ldots,N$. The network is then convergent.

This completes the proof. □

4. Simulation

In this section, we provide simulation results to illustrate and verify the theory developed.

Example 1. Consider a MVN neural network with two neurons, $K=6$

$$S(k+1)=CSIGN(WQS(k)), \tag{10}$$

where

$$W=\begin{bmatrix}0.1434 & -1.5743+0.0652i\\ -1.5743-0.0652i & -1.7375\end{bmatrix}.$$

Clearly, network (10) satisfies Theorem 1. We chose the initial value $S(0)=[0.7989+0.4966i-0.3725+0.1340i]^T$. By simulation, we can find that this network has two period-2 fixed points (Fig. 3): $S_1^*\approx[0.5+0.86603i, 0.5+0.86603i]^T$ and $S_2^*\approx[-0.5-0.86603i, -0.5-0.86603i]^T$.

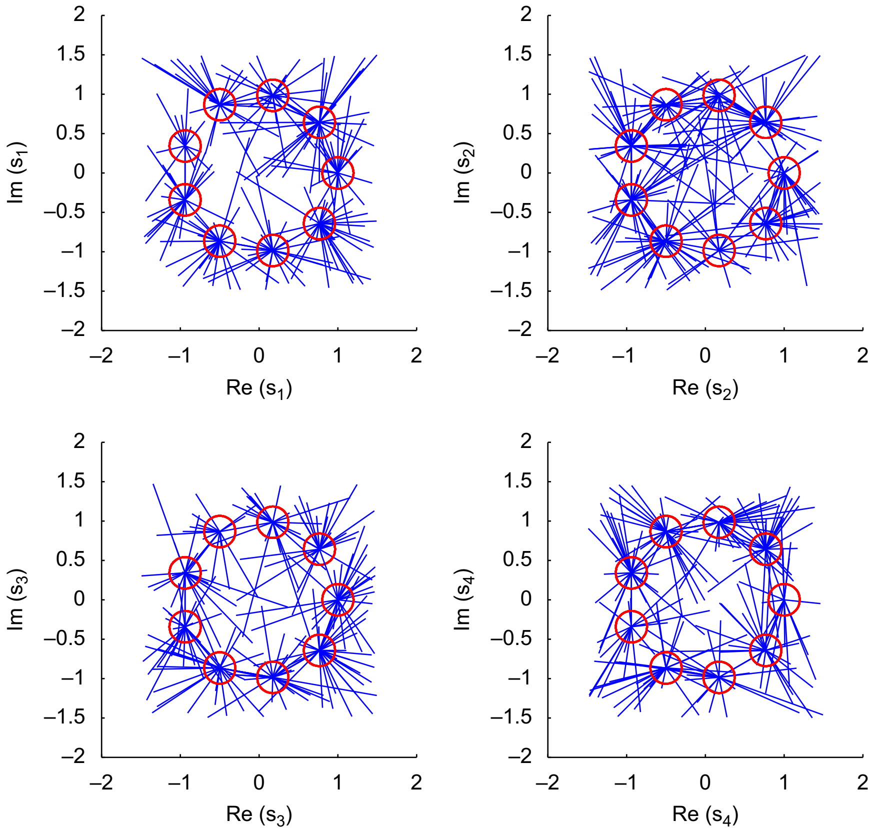

Fig. 5. Global attractivity of network (11) with $H=[0\ \ 0\ \ 0\ \ 0]^T$, the attractive regions are marked with a circle according to the activities of s_1–s_4.

Example 2. Consider a MVN neural network with two neurons, $K = 9$

$$S(k+1) = CSIGN(WQS(k) + QH), \tag{11}$$

where

$$W = \begin{bmatrix} 1.64 & -1.57+0.06i & -0.19-0.23i & -0.26-0.28i \\ -1.57-0.06i & 1.55 & -0.26+0.12i & -0.77+0.03i \\ -0.19+0.23i & -0.26-0.12i & 1.66 & -0.70+0.03i \\ -0.26+0.28i & -0.77-0.03i & -0.70-0.03i & 1.77 \end{bmatrix}$$

and

$$H = [1.1939 - 2.8638i - 1.1569 + 0.4717i - 1.3431 \\ - 1.8803i \ 1.1850 + 2.3114i]^T.$$

Network (11) satisfies Theorems 1 and 2, so it is completely stable. Fig. 4 shows the complete convergence of network (11) for 200 trajectories originating from randomly selected initial points in

$$Re(s(0)),\ Im(s(0)) \in [-1.5, 1, 5],$$

and we find at least nine fixed points from the simulation results.

Especially, when we set $H = [0\ 0\ 0\ 0]^T$, considering W is symmetric, we notice that almost all the states of one neuron can be a fixed point. Fig. 5 shows the complete convergence of network (11) with $H = [0\ 0\ 0\ 0]^T$ for 200 trajectories originating from randomly selected initial points in

$$Re(s(0)),\ Im(s(0)) \in [-1.5, 1, 5].$$

So, we can draw the conclusion that H is an important factor to determine the fixed point number of the system, which can also be got easily from the analysis on the relationship between $WQS(k)$ and QH in the complex-plane.

5. Conclusions

In this paper, we have investigated a class of discrete time recurrent neural networks with multivalued neurons in synchronous update mode. We also have established the network complete stability. Simulations carried out have validated theoretical findings. This paper focuses on the convergence study of such multi-stable networks, and may lead to design a associative memory based on such networks. Theorems 1 and 2 noted in our paper are just some convergence conditions, how to design and train such network is another interesting research topic related to further dynamics analysis. In short, we need to find stable and unstable fixed points, and store those useful patterns into those stable fixed points by adjusting weight. The discussion on network capacity is also necessary.

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